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LETTER TO THE EDITOR

A bosonic version of objects with fermion number $\frac{1}{2}$

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Abstract. Using the bosonisation procedure and a 'trial-and-error' method for general coupled non-linear field equations, we obtain a bosonic version of objects with fermion number $\frac{1}{2}$.

It is a well known fact nowadays that, in the presence of background fields with a non-trivial topological content, for example kinks in one-dimensional systems and monopoles in the three-dimensional ones, the fermionic number associated with the vacuum need not be integer and can even result in a transcendental function of the coupling constants of the theory [1]. Although the phenomenon, normally called fractionisation, can be interpreted physically according to different approaches (diagrammatic techniques, anomalies, Levinson's theorem), from a mathematical point of view it is related to the η invariant of the corresponding Dirac Hamiltonian [2]. This construct, introduced in the analysis of the index theorem for non-compact manifolds, represents a conveniently regularised expression of the operator's spectral asymmetry.

A simplified situation is reached supposing that the interaction admits a specific charge conjugation symmetry C, which relates the energy states $+|E|$ with those others of $-|E|$. It is understood that in these circumstances the physical interest of the problem, which in the general case spreads over all the spectrum, will only be concentrated in the zero-energy eigenstates. In fact, if they are normalisable, each one of them contributes $\frac{1}{2}$ to the fermionic number of the vacuum.

The theorem to be applied is the index theorem in open (non-compact) spaces, as first stated by Callias, and Bott and Seeley [3]. In this case the topological character of the phenomenon is obvious and shown already by observing that the corresponding index depends on the values that the background scalar field takes at infinity. For bidimensional systems the fractionisation in the presence of symmetry C has been related to typical SUSY quantum mechanics problems. In fact it is trivial to transform the initial Dirac equation for the spinor $\Psi(x)$ into a pair of Schrödinger type equations for the components $v(x)$ and $u(x)$. It happens then that the form of the equations enables us to interpret them as a simple SUSY quantum mechanics exercise, with the kink or classical solution $\phi_c(x)$ for the scalar field working as superpotential $W(x)$ [4]. Under these conditions the presence of zero modes can be detected by means of the Witten index $\Delta(\beta)$ [5], an object closely related to the spontaneous breaking of the SUSY.

With regards specifically to the $\Delta(\beta)$ determination, it is worth pointing out that in the presence of kinks, unlike for example what happens with $W(x)=x^n$ type superpotentials, an explicit dependence on the β regulator occurs; only in the limit in which β itself tends to infinity are the relevant physical results recovered [6]. Applying the Callias theorem, the limit to be considered is that which corresponds to taking the parameter z equal to zero [7].

For the models formulated in $1+1$ dimensions the study can be made easier using the so-called bosonisation method, both in its Abelian and non-Abelian versions. In the Abelian case the fermionic field is written as a non-local expression of the associated bosonic field, but certain bilinears are transformed in a strictly local way [8].

The aim set forth in this letter is to determine a bosonic version of the object whose fermionic number is $\frac{1}{2}$. To do this we will begin from the simple theory used by Jackiw and Rebbi [9], where they pointed out the possibility of fractionisation by means of a detailed analysis of the fermionic field. Here we will apply as an alternative the bosonisation procedure, in such a way that the corresponding study must be done on a two scalar coupled field system. Finally, and through a 'trial-and-error' technique [10], which in fact restricts the number of free constants, it is possible to reach the desired objective.

Let us begin with a $(\lambda\phi^4)_2$ theory governed by the following Lagrangian:

$$L=\int [\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{4}\lambda(\phi^2-m^2/\lambda)^2]\,\mathrm{d}x. \tag{1}$$

In the classical approach the homogeneous vacuum states are $\phi=\pm m/\lambda^{1/2}$; to introduce the kink it is enough to take an interpolation between $\phi=-m/\lambda^{1/2}$ at $x=-\infty$ and $\phi=m/\lambda^{1/2}$ for $x=\infty$.

Let us perform now a coupling to Dirac fermions in the typical Yukawa form:

$$L=\int [\tfrac{1}{2}(\partial_\mu\phi)^2+\bar{\Psi}\mathrm{i}\gamma^\mu\partial_\mu\Psi-\tfrac{1}{4}\lambda(\phi^2-m^2/\lambda)^2-g\phi\bar{\Psi}\Psi]\,\mathrm{d}x. \tag{2}$$

By means of the standard bosonisation techniques, with the arbitrary mass parameter M equal to the already available m, the Lagrangian will take the form [11]

$$L=\int [\tfrac{1}{2}(\partial_\mu\phi)^2+\tfrac{1}{2}(\partial_\mu\sigma)^2-\tfrac{1}{4}\lambda(\phi^2-m^2/\lambda)^2-gm\phi\cos 2\sqrt{\pi}\sigma]\,\mathrm{d}x \tag{3}$$

which describes a sine-Gordon system coupled to another scalar field. Once again from a classical point of view the homogeneous vacua are

$$\begin{aligned} &\phi=-a' \qquad \sigma=n\sqrt{\pi}\\ &\phi=a' \qquad \sigma=(n+\tfrac{1}{2})\sqrt{\pi} \end{aligned} \tag{4}$$

where the a' value just refers to the minimum of the function

$$F(\phi)=\tfrac{1}{4}\lambda(\phi^2-m^2/\lambda)^2-gm\phi. \tag{5}$$

To guarantee that indeed they are genuine minima we must look at the Hessian, which for either of the two possibilities (5) is reduced to

$$H=\begin{bmatrix} 3\lambda a'^2-m^2 & 0\\ 0 & 4\pi mga' \end{bmatrix}. \tag{6}$$

Curiously the condition which must be imposed on a' is to satisfy

$$a'>m/\sqrt{3\lambda} \tag{7}$$

just the point from which the effective potential calculated at one-loop order takes on complex values [12].

When considering how the formation of kinks not only the behaviour of ϕ must be specified but also prescriptions which refer to the new scalar field σ are necessary. A first possibility would consist in prescribing a transition of (ϕ, σ) from $(-a', 0)$ to $(-a', \pi^{1/2})$. If we remember that in terms of σ the fermionic number is [11]

$$Q_f = (1/\sqrt{\pi})[\sigma(\infty) - \sigma(-\infty)] \tag{8}$$

the previous case is included in the homogeneous vacuum case for ϕ, which as we know can only give rise to integer fermionic numbers.

The most interesting situation is represented by a system which interpolates from $(-a', 0)$ to $(a', \frac{1}{2}\pi^{1/2})$ and that naturally exhibits the fractionisation phenomenon. We have shown therefore the following fact: the vacuum fermionic charge, typical one-loop effect in the original version of the theory, can be reproduced in a particularly simple way on a classical level just by a previous bosonisation of the model.

We are now prepared to tackle the determination of a topological type kink with two coupled scalar fields and that in addition would support a fermionic number equal to $\frac{1}{2}$. It is a matter of finding a static solution, with the adequate boundary conditions, of the system

$$\begin{aligned} \mathrm{d}^2\phi/\mathrm{d}x^2 &= \lambda\phi(\phi^2 - m^2/\lambda) + gm\cos 2\sqrt{\pi}\sigma \\ \mathrm{d}^2\sigma/\mathrm{d}x^2 &= -gm2\sqrt{\pi}\phi \sin 2\sqrt{\pi}\sigma. \end{aligned} \tag{9}$$

Some examples of this same kind have been solved with more or less success, always bearing in mind that there is not a general procedure that resolves situations such as those. The method to be used here is suggested in [10], and it can be defined as a 'trial-and-error' technique which, although restricting the free parameters of the theory, is capable of providing at least some solutions.

With the potential of (3)

$$V(\phi, \sigma) = \tfrac{1}{4}\lambda(\phi^2 - m^2/\lambda)^2 + gm\phi\cos 2\sqrt{\pi}\sigma \tag{10}$$

if we look for a (ϕ, σ) solution which interpolates from $(-a', 0)$ to $(a', \frac{1}{2}\pi^{1/2})$ the most natural choice for the trial orbit function $g(\phi, \sigma)$ would be [10]

$$g(\phi, \sigma) = \phi + a'\cos 2\sqrt{\pi}\sigma = 0. \tag{11}$$

Taking into account that

$$\partial g/\partial\phi = 1 \qquad \partial g/\partial\sigma = -a'2\sqrt{\pi}\sin 2\sqrt{\pi}\sigma \tag{12}$$

and

$$\partial V/\partial\phi = \lambda\phi(\phi^2 - m^2/\lambda) + gm\cos 2\sqrt{\pi}\sigma \qquad \partial V/\partial\sigma = -gm\phi 2\sqrt{\pi}\sin 2\sqrt{\pi}\sigma \tag{13}$$

we can write [10]

$$\begin{aligned} &\int_{-a'}^{\phi} (\lambda\tilde{\phi} - m^2\tilde{\phi} + gm\cos 2\sqrt{\pi}\tilde{\sigma})\,\mathrm{d}\tilde{\phi} \\ &\qquad = a'^2 4\pi \sin^2 2\sqrt{\pi}\sigma \int_0^{\sigma} (-gm2\sqrt{\pi}\tilde{\phi}\sin 2\sqrt{\pi}\tilde{\sigma})\,\mathrm{d}\tilde{\sigma}. \end{aligned} \tag{14}$$

Using now the trial orbit marked by (11) the previous integral is converted into

$$\int_{-a'}^{\phi} (\lambda\tilde{\phi}^3 - m^2\tilde{\phi} - (gm/a')\tilde{\phi})\,\mathrm{d}\tilde{\phi}$$

$$= a'^2 4\pi \sin^2 2\sqrt{\pi}\sigma \int_0^{\sigma} gm\sqrt{\pi}a' \sin 4\sqrt{\pi}\tilde{\sigma}\,\mathrm{d}\tilde{\sigma} \tag{15}$$

and we obtain easily

$$\tfrac{1}{4}\lambda\phi^4 - (m^2 + gm/a')\phi^2 = \tfrac{1}{4}(4\pi gm/a')\phi^4 - \tfrac{1}{2}(a'4\pi gm)\phi^2. \tag{16}$$

The pertinent identifications would then be

$$\lambda = 4\pi gm/a' \qquad m^2 + gm/a' = a'4\pi gm. \tag{17}$$

Let us return now specifically to the solution of the differential equations which correspond to ϕ and σ, since having determined an orbit they uncouple and so the integration is feasible; in fact

$$\mathrm{d}^2\phi/\mathrm{d}x^2 = \lambda\phi^3 - (m^2 + gm/a')\phi. \tag{18}$$

Calling $m'^2 = m^2 + gm/a'$ it is trivial to obtain

$$\phi(x) = a' \tanh \frac{m'}{\sqrt{2}} x. \tag{19}$$

Regarding the σ-field bosonised version of the Ψ, the equation is

$$\mathrm{d}^2\sigma/\mathrm{d}x^2 = gm\sqrt{\pi}a' \sin 4\sqrt{\pi}\sigma. \tag{20}$$

The integration is carried out by bearing in mind that it is a typical sine-Gordon problem

$$\sigma(x) = \frac{1}{\sqrt{\pi}} \tan^{-1}[\exp(2\sqrt{gm\pi a'}x)]. \tag{21}$$

In view of (19) and (21) the essentially topological character of the kink is seen; it suffices to observe the values of the fields (ϕ, σ) at $x = -\infty$ and $x = \infty$. Moreover the fermionic number which exhibits the solution found is

$$Q_{\mathrm{f}} = \frac{1}{\sqrt{\pi}}[\sigma(\infty) - \sigma(-\infty)] = \tfrac{1}{2}. \tag{22}$$

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